Error Budgets, and Introduction to Class Projects

Lecture 6, ASTR 289

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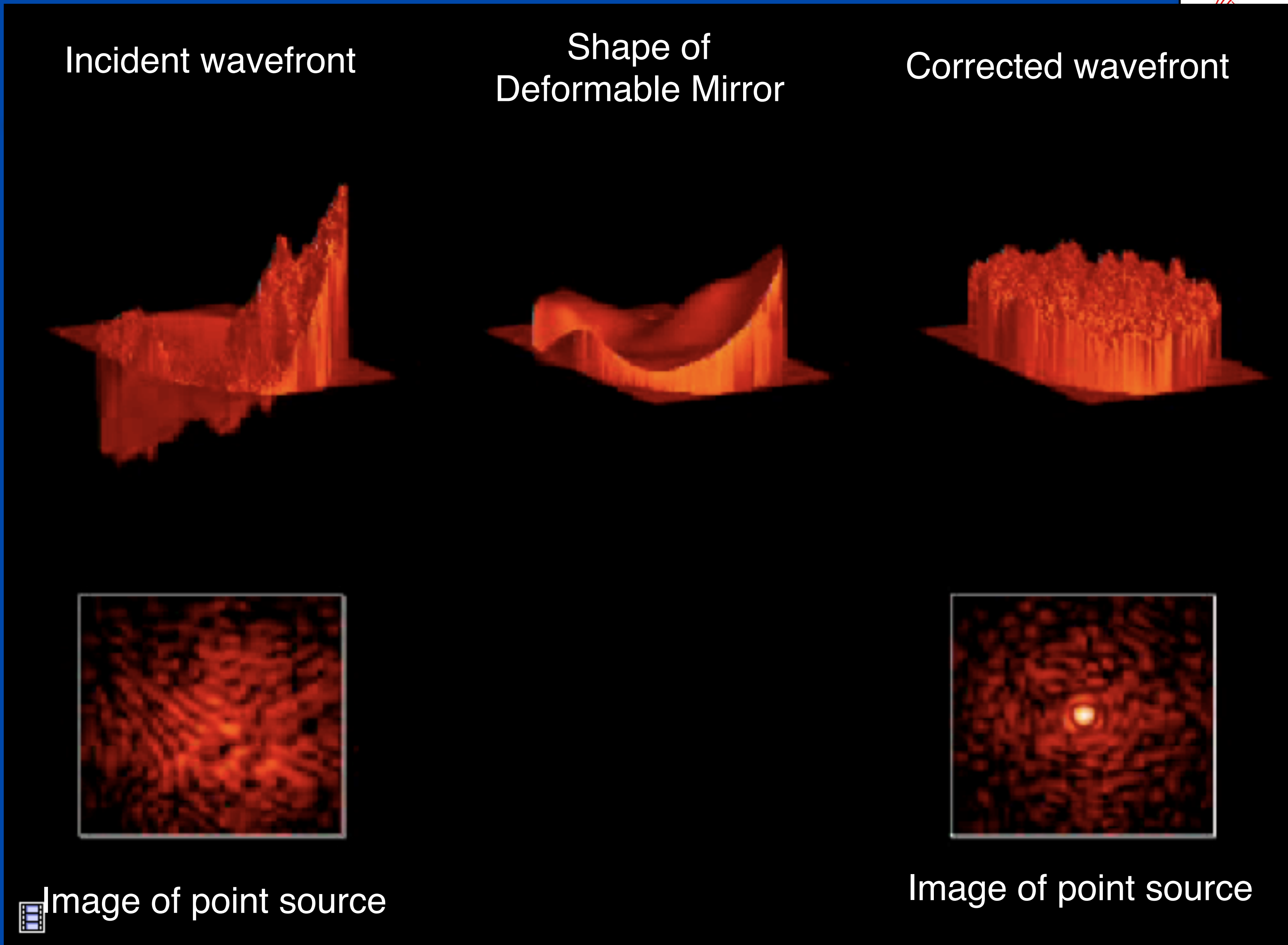
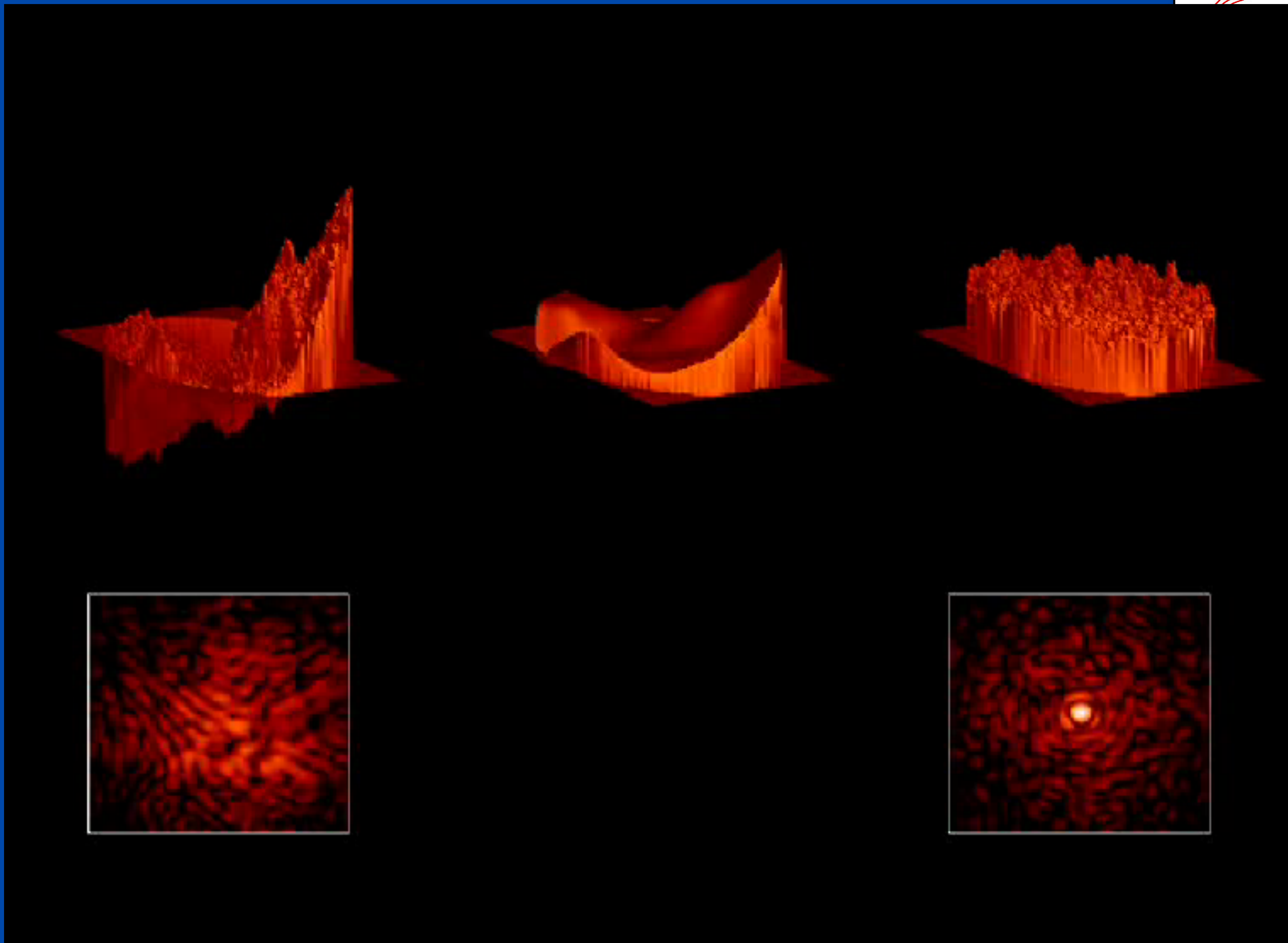
What is "residual wavefront error"?

Very distorted wavefront

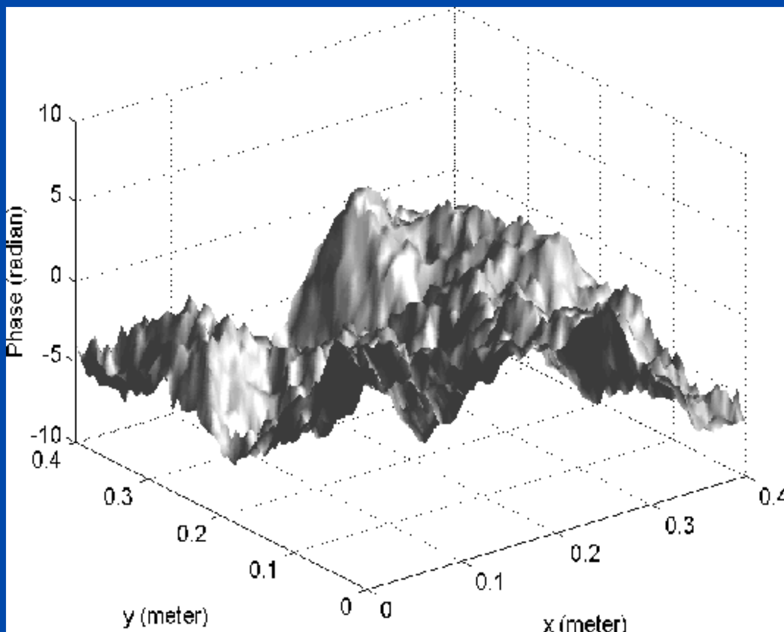
Less distorted wavefront (but still not perfect)



Credit: James Lloyd, Cornell Univ.



How to calculate residual wavefront error



- Optical path difference = *Δz* where *k Δz* is the phase change due to turbulence
- Phase variance is $\sigma^2 = \langle (k\,\Delta z)^2 \rangle$
- If several independent effects cause changes in the phase,

$$\sigma_{tot}^2 = k^2 \left\langle \left(\Delta z_1 + \Delta z_2 + \Delta z_3 + \ldots \right)^2 \right\rangle$$

$$= k^2 \left\langle \left(\Delta z_1 \right)^2 + \left(\Delta z_2 \right)^2 + \left(\Delta z_3 \right)^2 + \ldots \right\rangle$$

- Sum up the contributions from individual physical effects independently

An error budget can describe wavefront phase or optical path difference

$$\sigma_{tot}^2 = k^2 \left\langle \left(\Delta z_1 + \Delta z_2 + \Delta z_3 + \ldots \right)^2 \right\rangle$$

$$= k^2 \left\langle \left(\Delta z_1 \right)^2 + \left(\Delta z_2 \right)^2 + \left(\Delta z_3 \right)^2 + \ldots \right\rangle$$

- Be careful of units (Hardy and I will both use a variety of units in error budgets):
  - For tip-tilt residual errors: variance of tilt angle $\langle\alpha^2\rangle$
  - Optical path difference $n\Delta z$ in meters: OPD_m
  - Optical path difference $n\Delta z$ in waves: $OPD_\lambda = \Delta z / \lambda$
  - Optical path difference in radians of phase:
    - $\varphi = 2\pi \, OPD_\lambda = (2\pi/\lambda) \, OPD_m = k \, OPD_m$

Units of wavefront error

- Electromagnetic wave propagation

$$E = E_0 \exp(i\phi) = E_0 \exp i(kx - \omega t) = E_0 \exp i\left(\frac{2\pi n x}{\lambda} - \omega t\right)$$

- Change in phase due to variation in index of refraction *n*
- Can express as:
  - Phase $\Phi \sim k\Delta x = k_0 n \Delta x$ (units: radians)
  - Optical path difference $\Phi / k = \Delta x$ (units: length)
    - Frequently microns or nanometers
  - Waves: $\Delta x / \lambda$ (units: dimensionless)

Question

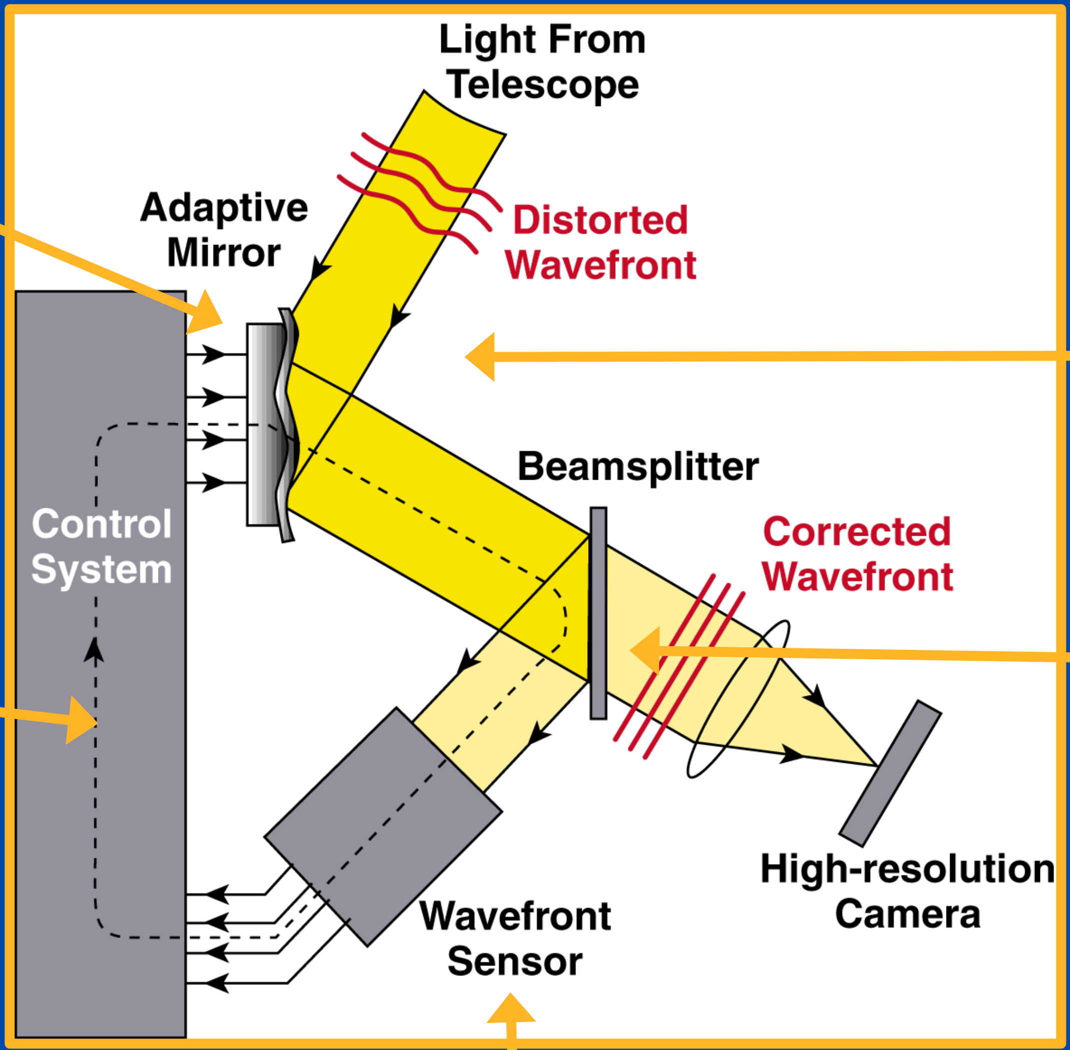
If the total wavefront error is
$$\sigma_{tot}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \ldots$$

- List as many physical effects as you can think of that might contribute to the overall wavefront error σ_{tot}^2

Elements of an adaptive optics system

Deformable mirror fitting error

temporal delay, noise propagation



Not shown: tip-tilt error, anisoplanatism error

Non-common path errors

Measurement error

What is an “error budget” ?

1. The allocation of statistical variations and/or error rates to individual components of a system, in order to satisfy the full system's end-to-end performance specifications.
2. The term “error budget” is a bit misleading: it doesn’t mean “error” as in “mistake” - it means performance uncertainties, and/or the imperfect, “real life” performance of each component in the system.
3. In a new project: Start with “top down” performance requirements from the science that will be done. Allocate “errors” to each component to satisfy overall requirements. As design proceeds, replace “allocations” with real performance of each part of system. Iterate.

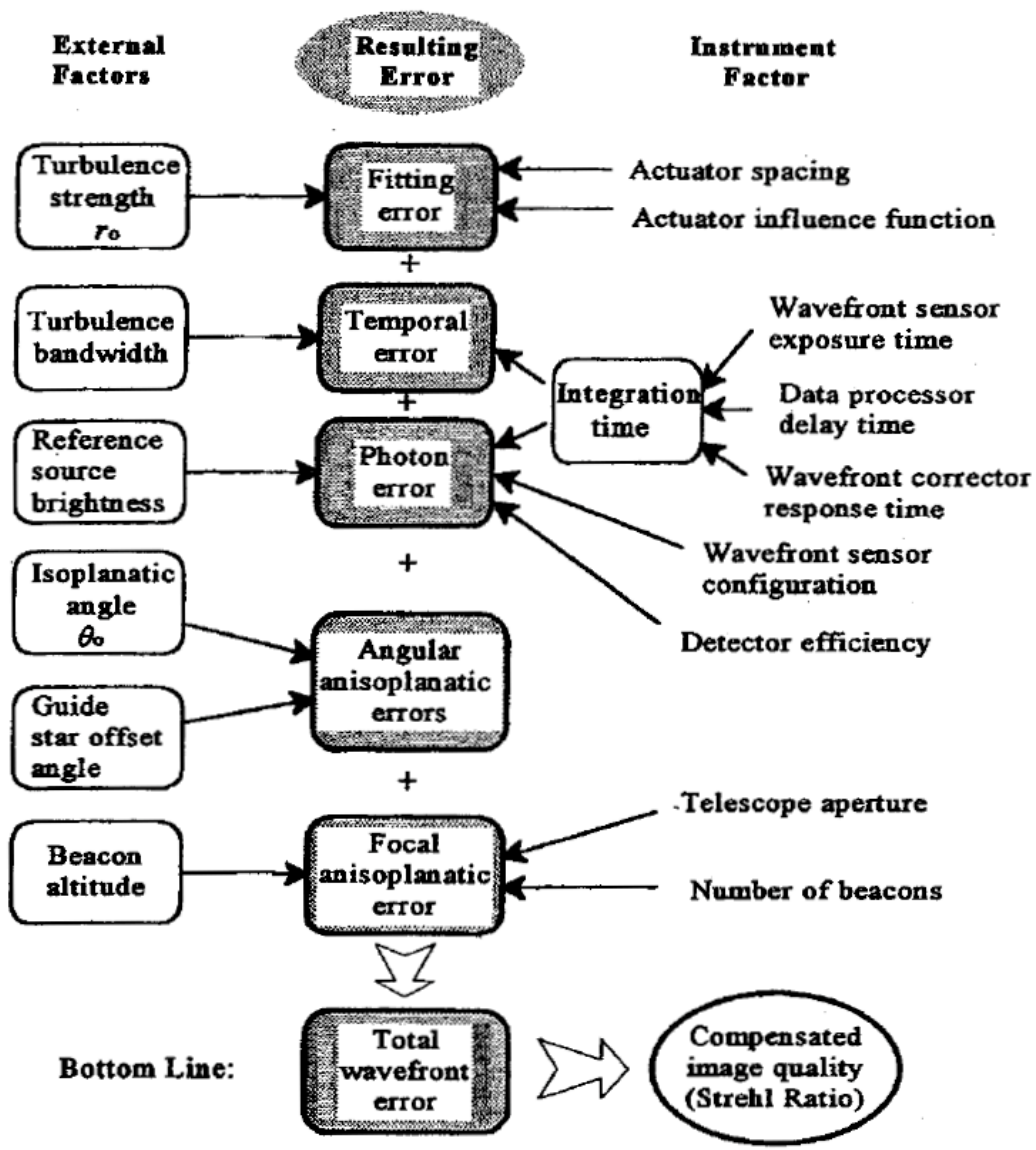


Figure 2.32 Main sources of wavefront error in adaptive optics.

Hardy
Figure 2.32

Wavefront errors due to time lags, τ_0

- Wavefront phase variance due to $\tau_0 = f_G^{-1}$
  - If an AO system corrects turbulence "perfectly" but with a phase lag characterized by a time τ, then

$$\sigma_\tau^2 = 28.4 \, (\tau/\tau_0)^{5/3}$$

Hardy Eqn 9.57

- The factor of 28.4 out front is significant penalty: have to run your AO system faster than $\tau = \tau_0$
- For $\sigma_\tau^2 < 1$, $\tau < 0.13\ \tau_0$
- In addition, closed-loop bandwidth is usually ~ 10x sampling frequency ➔ have to run even faster

Wavefront variance due to isoplanatic angle θ_0

- If an AO system corrects turbulence "perfectly" but uses a guide star an angle θ away from the science target, then

$$\sigma^2_{angle} = \left(\frac{\theta}{\theta_0} \right)^{5/3}$$

Hardy Eqn 3.104

- Typical values of θ_0 are a few arc sec at λ = 0.5 μm, 15-20 arc sec at λ = 2 μm

Deformable mirror fitting error

- Accuracy with which a deformable mirror with subaperture diameter *d* can remove wavefront aberrations
- With a finite number of actuators, you can't do a perfect fit to an arbitrary wavefront

$$\sigma_{DM\_fitting} = \mu \left(\frac{d}{r_0} \right)^{5/3}$$

- Constant μ depends on specific design of deformable mirror
- For segmented mirror that corrects tip, tilt, and piston (3 degrees of freedom per segment) μ = 0.14
- For deformable mirror with continuous face-sheet, μ = 0.28

Error budget concept (sum of σ^2 's)

$$\sigma_{tot}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \ldots \ \text{radians}^2$$

- There's not much to be gained by making any particular term much smaller than all the others: try to roughly equalize all the terms

- Individual terms we know so far:
  - Anisoplanatism $\sigma_{angle}^2 = (\theta/\theta_0)^{5/3}$
  - Temporal error $\sigma_\tau^2 = 28.4 \ (\tau/\tau_0)^{5/3}$
  - Fitting error $\sigma_{DM\_fitting} = \mu(d/r_0)^{5/3}$

We will discuss other wavefront error terms in coming lectures

- Tip-tilt errors
  - Imperfect sensing and/or correcting of image wander, due to atmosphere or telescope motions (wind shake)
- Measurement error
  - Wavefront sensor doesn't make perfect measurements
  - Finite signal to noise ratio, optical limitations, …
- Non-common-path errors
  - Calibration of different optical paths between science instrument and wavefront sensor isn't perfect
- Calibration errors
  - What deformable mirror shape would correspond to a perfectly flat wavefront?

Error budget so far

$$\sigma^2_{tot} = \sigma^2_{fitting} + \sigma^2_{anisopl} + \sigma^2_{temporal} + \sigma^2_{meas} + \sigma^2_{calib} + \sigma^2_{tip-tilt} + ...$$

√ √ √

Still need to work these out

Try to "balance" error terms: if one is big, no point struggling to make the others tiny

Keck AO error budget example (not current)

Error Term (nm)	Predicted	Measured
DM: Atmospheric fitting error	110	139
DM: Telescope fitting error	66	60
Calibration (non-common path)	114	113
Finite Bandwidth (high order)	115	103
WFS measurement error*	0	0
TT bandwidth	91	75
TT measurement	5	9
Miscellaneous	106	120
Total wavefront error	249	258
K-band Strehl	0.60	0.58

* Very bright star

Assumptions:

Natural guide star is very bright (no measurement error)

10 degree zenith angle

Wavefront sensor bandwidth: 670 Hz

Note that uncorrectable errors in telescope itself are significant

We want to relate phase variance $\langle\sigma^2\rangle$ to the "Strehl ratio"

- Two definitions of Strehl ratio (equivalent):

1. Ratio of the maximum intensity of a point spread function to what the maximum would be without any aberrations:
$$S \equiv \left(I_{\max} / I_{\max\_no\_aberrations} \right)$$

2. The "normalized volume" under the optical transfer function of an aberrated optical system

$$S \equiv \frac{\int OTF_{aberrated}(f_x, f_y)\, df_x\, df_y}{\int OTF_{un-aberrated}(f_x, f_y)\, df_x\, df_y}$$

$$\text{where } OTF(f_x, f_y) = Fourier\ \ Transform(PSF)$$

Relation between phase variance and Strehl ratio

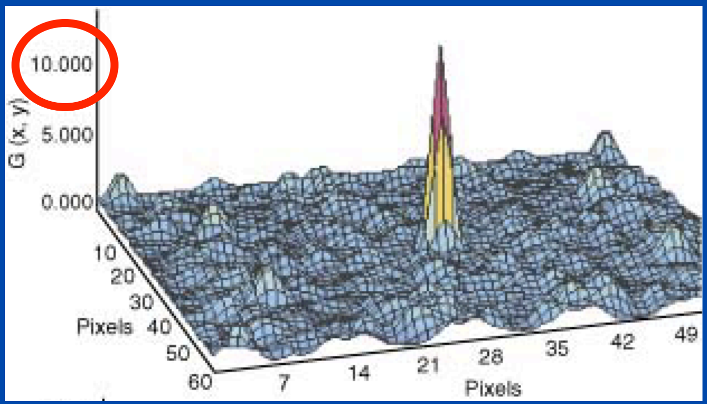
- "Maréchal Approximation"

$$\text{Strehl} \cong \exp\left(-\sigma_\phi^2\right)$$

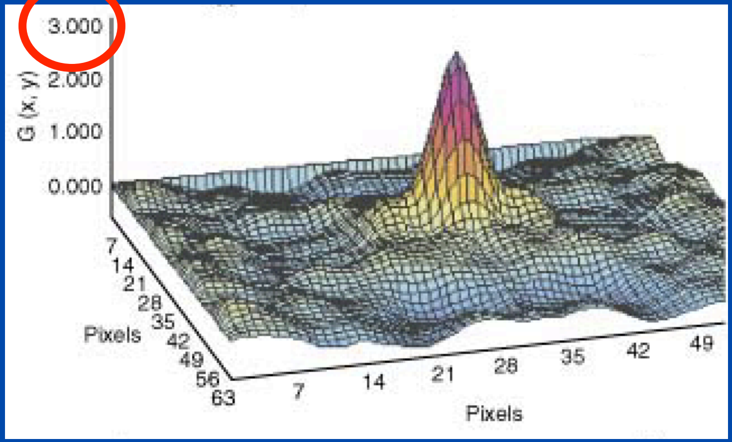
where $\sigma_\varphi{}^2$ is the total wavefront variance

- Valid when Strehl > 10% or so ($\sigma_\varphi{}^2 < 2.3$)
- Under-estimates Strehl for larger values of $\sigma_\varphi{}^2$

High Strehl ⇒ PSF with higher peak intensity and narrower "core"

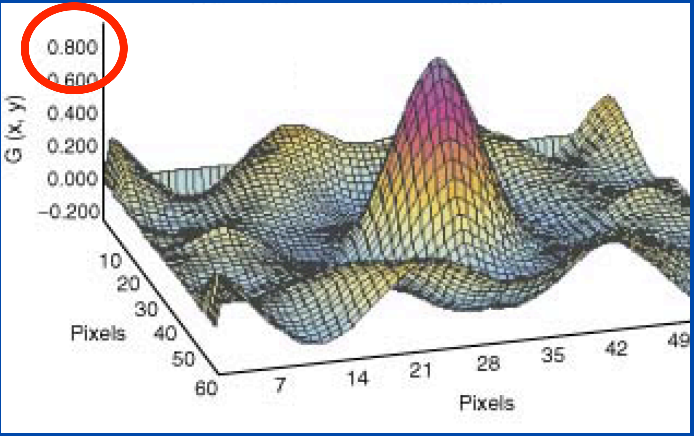


High Strehl



Medium Strehl

Low Strehl



Characterizing image motion or tip-tilt

Image motion or "tip-tilt" also contributes to total wavefront error

- Turbulence both blurs an image and makes it move around on the sky (image motion).
  - Due to overall "wavefront tilt" component of the turbulence across the telescope aperture

Angle of arrival fluctuations $\langle \alpha^2 \rangle = 0.364 \left(\frac{D}{r_0} \right)^{5/3} \left(\frac{\lambda}{D} \right)^2 \propto \lambda^0 \ \mathrm{D}^{-1/3} \ \text{radians}^2$

(Hardy Eqn 3.59 - one axis)

- Can "correct" this image motion using a tip-tilt mirror (driven by signals from an image motion sensor) to compensate for image motion

Scaling of tip-tilt with λ and D: the good news and the bad news

- In absolute terms, rms image motion in radians is <u>independent</u> of λ, and decreases slowly as D increases:

$$\left\langle \alpha^2 \right\rangle^{1/2} = 0.6 \left(\frac{D}{r_0} \right)^{5/6} \left(\frac{\lambda}{D} \right) \quad \propto \; \lambda^0 \; \mathrm{D}^{-1/6} \; \text{radians}$$

- But you might want to compare image motion to diffraction limit at your wavelength:

$$\frac{\left\langle \alpha^2 \right\rangle^{1/2}}{(\lambda / D)} \sim \frac{\mathrm{D}^{5/6}}{\lambda}$$

Now image motion <u>relative to diffraction limit</u> is almost ~ D, and becomes larger fraction of diffraction limit for smaller λ

Long exposures, no AO correction

$$FWHM(\lambda) = 0.98 \frac{\lambda}{r_0}$$

- "Seeing limited": Units are radians
- Seeing disk gets slightly smaller at longer wavelengths:

  $$\text{FWHM} \sim \lambda / \lambda^{-6/5} \sim \lambda^{-1/5}$$

- For completely uncompensated images, wavefront error

  $$\sigma^2_{uncomp} = 1.02 \ (D \ / \ r_0 \ )^{5/3}$$

Correcting tip-tilt has relatively large effect, for seeing-limited images

- For completely uncompensated images

$$\sigma^2_{uncomp} = 1.02 \; (D \;/\; r_0 \;)^{5/3}$$

- If image motion (tip-tilt) has been completely removed

$$\sigma^2_{tiltcomp} = 0.134 \; (D \;/\; r_0 \;)^{5/3}$$

(Tyson, Principles of AO, eqns 2.61 and 2.62)

- Removing image motion can (in principle) improve the wavefront variance of an uncompensated image by a factor of 10

- Origin of statement that **"Tip-tilt is the single largest contributor to wavefront error"**

But you have to be careful if you want to apply this statement to AO correction

- If tip-tilt has been <u>completely</u> removed

$$\sigma^2_{tiltcomp} = 0.134\ (\ D\ /\ r_0\ )^{5/3}$$

- But typical values of $(\ D\ /\ r_0\ )$ are 10 - 50 in the near-IR
  - Keck, D=10 m, r_0 = 60 cm at λ=2 μm, $(\ D/r_0\ )$ = 17

$$\sigma^2_{tiltcomp} = 0.134\ (\ 17\ )^{5/3}\ \sim\ 15$$

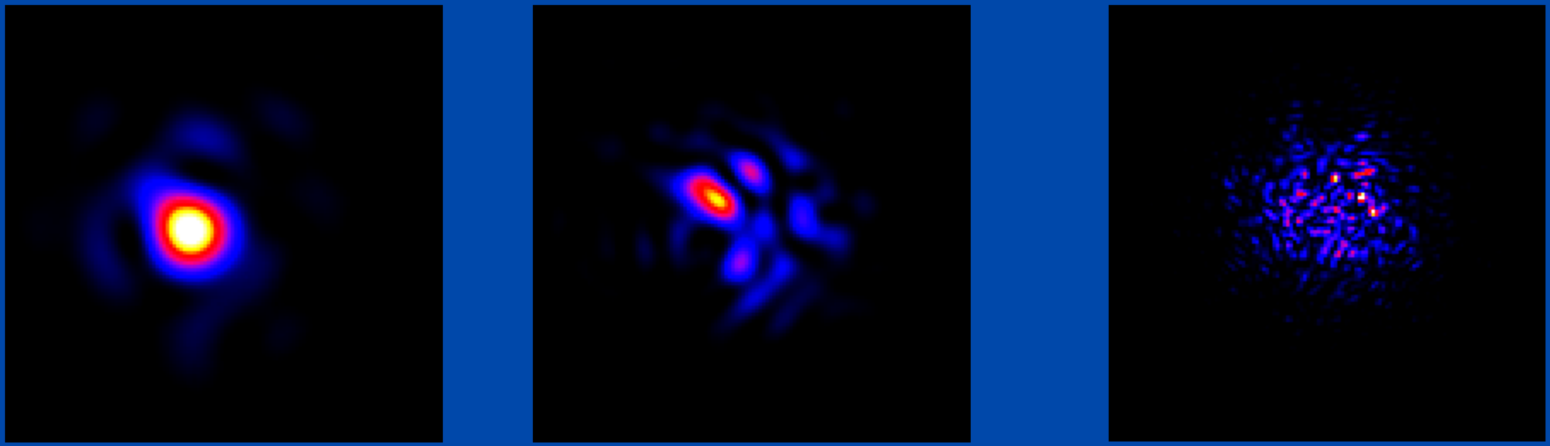
so wavefront phase variance is >> 1

- Conclusion: if $(\ D/r_0\ )\ \gg 1$, removing tilt alone won't give you anywhere near a diffraction limited image

Effects of turbulence depend on size of telescope

- Coherence length of turbulence: r_0 (Fried's parameter)
- For telescope diameter D < (2 - 3) x r_0 :
  Dominant effect is "image wander"
- As D becomes >> r_0 :
  Many small "speckles" develop

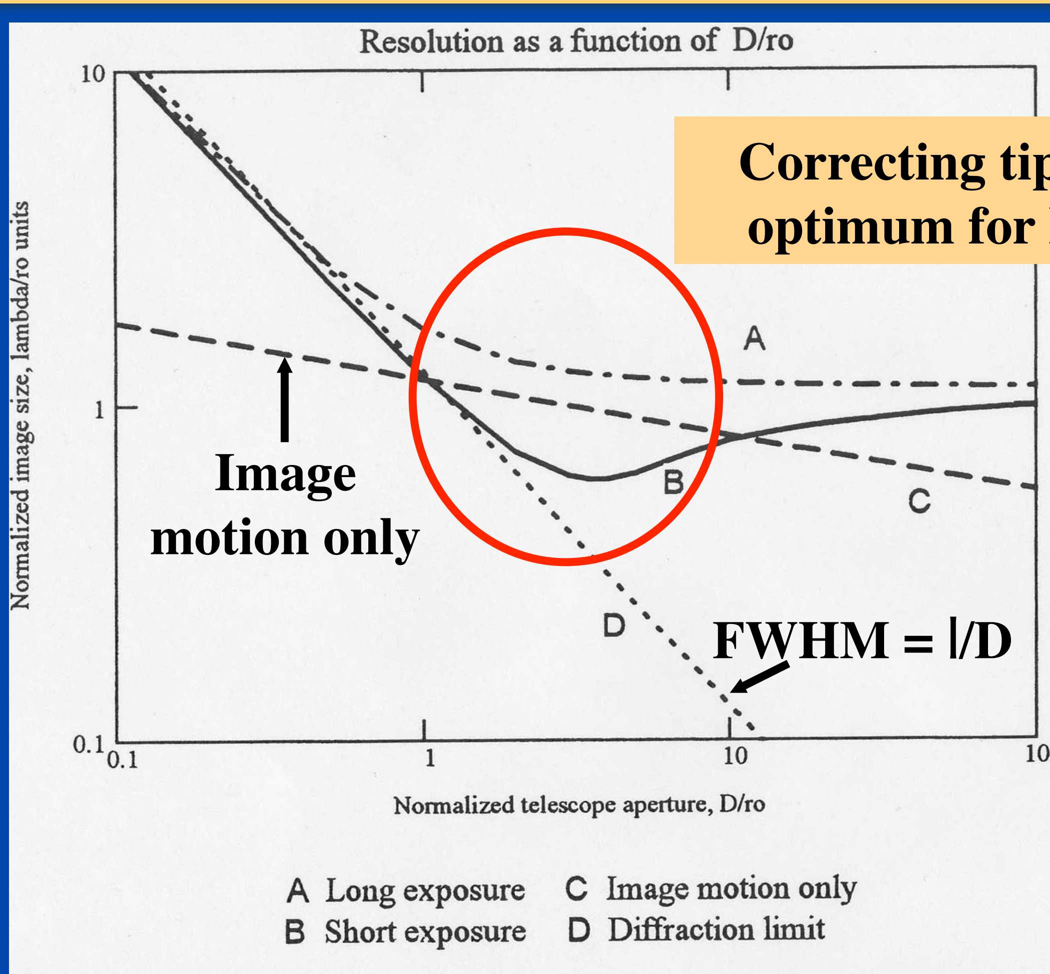
- Computer simulations by Nick Kaiser: image of a star, r_0 = 40 cm



D = 1 m **D = 2 m** **D = 8 m**

Effect of atmosphere on long and short exposure images of a star

Hardy p. 94



Correcting tip-tilt <u>only</u> is optimum for $D/r_0 \sim 1 - 3$

Vertical axis is image size in units of λ / r_0

Summary of topics discussed today

- Wavefront errors due to:
  - Timescale of turbulence
  - Isoplanatic angle
  - Deformable mirror fitting error
  - Other effects
- Concept of an "error budget"
- Image motion or tip-tilt
- Goal: to calculate $\langle\sigma_\varphi^2\rangle$ and thus the Strehl ratio

$$\mathrm{Strehl} \cong \exp\left(-\sigma_\phi^2\right)$$